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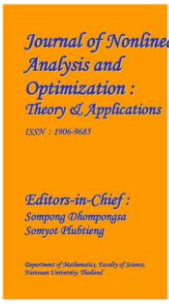

A NEW TYPE OF SEPARATION AXIOMS IN TOPOLOGICAL SPACES USING pgrβ OPEN SETS

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Abstract

Here we create a novel class of separation axioms based on pgrβ-open sets in topological spaces, and we explore some of its characterizations and connections.

Key Words pgrβ-T_0, pgrβ-T_1, pgrβ-T_2 spaces, pgrβ-D set, pgrβ-D_0, pgrβ-D_1, pgrβ-D_2 spaces

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1. Introduction and Preliminaries

The key ideas in topological spaces that can be utilized to construct more limited classes of topological spaces are separation axioms. Following the development and exploration of generalized closed sets, which were initially stated in general topology by Levine [1], many separation axioms were developed, because most weak separation axioms are stated in terms of generalized closed sets. Frechet created the T_1-axiom in 1907. T_2 axiom was published by Hausdorff in 1923, and Kolmogorov published T_0 axiom in 1933. Tong [4] presented the concepts of the D-type separation axioms in 1982. Open sets were used to present the idea of D sets, and the idea of those D sets was then utilized to define D spaces. Using the idea of pgrβ-open sets introduced in [3], we present a new type of separation axioms called pgrβ-separation axioms. We also examine the ideas of pgrβ-T_0, pgrβ-T_1, pgrβ-T_2 spaces, pgrβ-D set, pgrβ-D_0, pgrβ-D_1 and pgrβ-D_2 spaces. Additionally, we look at various descriptions of these areas and their connections to one another. Here a topological space with the notation (G,μ) is one for which no separation axiom is necessary. The terms cl(A) and int(A), respectively, represent the closure of A and the interior of A for a subset A of this topological space (G,μ),while pgrβ-O(G) and pgrβ-C(G), designate all pgrβ-open set families of G and all pgrβ-closed set families of G, respectively.

2. Separation axioms using pgrβ–open sets in topological spaces

Definition 2.1 : pgrβ-T_k Spaces (k = 0, 1, 2) : A topological space G is called

(i) a pgrβ-T_0 space if for a set of two separate elements a, b of G, there is a pgrβ-open set M in G having just one of a and b, but not both,

(ii) a pgrβ-T_1 space if for a set of two separate elements a, b of G, there are two pgrβ-open sets M and N in G where M has a but no b and N has b but no a

(iii) a pgrβ-T_2 space if for a set of two separate elements a, b of G, there are distinct pgrβ-open sets K and L where K has a but no b and L has b but no a.

Theorem 2.2:

(i) Every T_0 space is a pgrβ-T_0 space

(ii) Every T_1 space is a pgrβ-T_0 space.

(iii) Every T_1 space is a pgrβ-T_1 space.
(iv) Every T_2 space is a pgrβ-T_2 space.
Proof: Straight forward from the definition 3.1.
The converse of the theorem need not hold, as shown by the examples that follow.

Example 2.3: Let G ={1, 2, 3} and μ ={∅,G,{1},{1, 3}}.Then pgrβ-O(G) ={∅,G,{1},{2}, {3}, {1, 2},{1, 3}}. Here (G, μ) is pgrβ-T_0 space but not T_0 space.

Example 2.4: Let G ={1, 2, 3} and μ ={∅,G,{1, 2}}.Then pgrβ-O(G) ={G, ∅,{ 1},{2},{3}, {1, 2},{1, 3},{2, 3}}. Here (G, μ) is pgrβ-T_0 space but not T_1 space.

Example 2.5: Let G ={1, 2, 3} and μ ={ ∅,G,{3},{1, 3},{2, 3}}.Then pgrβ-O(G) ={G, ∅,{1}, {2},{3},{1, 3},{2, 3}}. Here (G, μ) is pgrβ-T_1 space but not T_1 space.

Example 2.6: Let G ={1, 2, 3} and μ ={∅,G,{1},{2},{1, 2}}. Then pgrβ-O(G) ={∅,G,{1},{2}, {3},{1, 2},{1, 3},{2, 3}}. Here (G, μ) is pgrβ-T_2 space but not T_2 space.

Theorem 2.7: (i) All pgrβ-T_1 spaces are pgrβ-T_0 spaces.
(ii) All pgrβ-T_2 spaces are pgrβ-T_1 spaces.
Proof: Straight forward.
The converse of the theorem need not hold, as shown by the examples that follow.

Example 2.8: Let G ={1, 2, 3} and μ ={ ∅,G,{1},{2},{1, 2},{2, 3}}.Then pgrβ-O(G) ={∅,G, {1},{2},{1, 2},{2, 3}}. Here (G, μ) is pgrβ-T_0 space but not pgrβ-T_1 space.

Example 2.9: Let G ={1, 2, 3} and μ ={ ∅,G,{3},{1, 3},{2, 3}}.Then pgrβ-O(G) ={G, ∅,{1}, {2},{3},{1, 3},{2, 3}}. Here (G, μ) is pgrβ-T_1 space but not pgrβ-T_2 space.

Theorem 2.10: Every subspace of a pgrβ-T_0 space is a pgrβ-T_0 space.
Proof: Consider H be a subspace of a pgrβ-T_0 space G. Let a and b be set of two separate elements of H. As H is a subspace of G, the elements a and b are also distinct in G. Since G is pgrβ-T_0 space, there is a pgrβ-open set M whereM has a but no b. Then H ∩M is pgrβ-open in H with a but no b. Hence H is pgrβ-T_0 space.

Definition 2.11: The pgrβ-closure (resp. pgrβ-interior) of A in G is the intersection (resp. union) of all pgrβ-closed(resp. pgrβ-open) sets that include (resp. are contained in) a set A in G and it is indicated as pgrβ-cl(A) (resp. pgrβ-int(A)).
Remark 2.12: Every pgrβ-closed set contains pgrβ-cl(A) which includes A for any set A in G.
Theorem 2.13: Let A ⊂ B where A and B are in G then
(i) pgrβ-cl(A) is the smallest pgrβ-closed set that includes A.
(ii) pgrβ-cl(A) ⊂ pgrβ-cl(B).
(iii) A is a pgrβ-closed set imply and implies pgrβ-cl(A) = A.
(iv) pgrβ-cl(pgrβ-cl(A)) = pgrβ-cl(A).
Theorem 2.14: The topological space (G, μ) is a pgrβ-T_0 space if and only if for each set of two separate points a, b of G, pgrβ-cl({a}) ≠ pgrβ-cl({b}).
Proof: Let (G, μ) be a pgrβ-T_0 space. If a, b ∈G such that a ≠ b, then, there is a pgrβ-open setV such that a ∈V and b ∉V. Then V^c is a pgrβ-closed with b but no a. But pgrβ-cl({b}) is the small among all pgrβ-closed sets with b. Therefore pgrβ-cl({b}) ⊂V^c and hence a ∉ pgrβ-cl({b}). Thus pgrβ-cl({a}) ≠ pgrβ-cl({b}).

In contrast, imagine a, b ∈ G, a ≠ b and pgrβ-cl({a}) ≠ pgrβ-cl({b}). Let c ∈ G such that c ∈ pgrβ-cl({a}) but c∉ pgrβ-cl({b}). If a ∈ pgrβ-cl({b}) then pgrβ-cl({a}) ⊂pgrβ-cl({b}), as a result c ∈ pgrβ-cl({b}). This contradicts itself. So a ∉ pgrβ-cl({b}) and this implies a ∈ $(\text{pgr}\beta\ \text{cl}(b))^c$. This shows $(\text{pgr}\beta\text{-cl}(\{b\}))^c$ is a pgrβ-open set with a but no b. Hence ((G, μ)) is pgrβ-T_0 space.
Theorem 2.15: A topological space G is pgrβ-T_1 space imply and implies for every a ∈ G, {a} is pgrβ-closed set in G.
Proof: Assume a is in pgrβ-T_1 space G. For{a} to be a pgrβ-closed set, we must demonstrate G -{a} is pgrβ-open set in G. Let b ∈G -{a}, implies a ≠ b ∈ G and since G is pgrβ-T_1 space. Then there are two pgrβ-open sets M, N where a ∉M, b ∈N ⊆ G -{a}.Since b ∈ N ⊆ G -{a} then G -{a} is pgrβ-

open set. Hence {a} is pgrβ-closed set. In contrast, let a ≠ b ∈ G then {a}, {b} are pgrβ-closed sets. That is G -{a} is pgrβ-open set. Clearly, a ∉ G -{a} and b ∈ G -{a}.Similarly G -{b} is pgrβ-open set, b ∉ G -{b} and a ∈ G -{b}. Hence G is pgrβ-T_1 space.

3. pgrβ– D_i Spaces (i = 0, 1, 2)

Definition 3.1: A nonempty subset A of a topological space (G, μ) is termed as pgrβ- Difference set in G (briefly pgrβ-D set) if there are two pgrβ-open sets G called U and V where U ≠ G and A =U /V.

Example 3.2: Let G ={1, 2, 3} further to the topology μ ={ ∅, G,{2},{2, 3}}. Here pgrβ-O(G) ={ ∅, G,{ 1},{2},{3},{1, 2},{1, 3}}. Then {{1},{2},{3},{1, 2},{1, 3}} are pgrβ-D sets in G.

Remark 3.3: Every pgrβ-open set U ≠ G in G is a pgrβ-D set in G, but not the other way around.

Example 3.4: Let G ={1, 2, 3, 4} further to the topology μ = { ∅,G,{1},{2, 4},{1, 2, 4}}. Here pgrβ-O(G) = { ∅, G, {1},{2},{4},{1, 2},{1, 4},{2, 4},{1, 2, 3},{1, 2, 4},{1, 3, 4}}. Consider U ={1, 2, 3} ≠ G and V ={1, 2, 4}. Then A = U /V ={3} is a pgrβ-D set in G but not a pgrβ-open set in G.

Definition 3.5: A topological space G is termed as a

(i) pgrβ-D_0 space if for each set of two separate elements a, b ∈G there is a pgrβ-D set of G with a but no b or a pgrβ-D set of G without a but with b.

(ii) pgrβ-D_1 space if for each set of two separate elements a, b ∈G having a ≠ b there is a pgrβ-D set of G with a but no b and a pgrβ-D set of G without a but with b.

(iii) pgrβ-D_2 space if for each set of two separate elements a, b ∈G having a ≠ b there are disjoint pgrβ-D sets E and F such that a ∈ E and b ∈ F.

Theorem 3.6: The space G possesses the characteristics listed below.

(i) If G is pgrβ-T_i space, then it is pgrβ-D_i space for i = 0,1,2

(ii) If G is pgrβ-D_i space, then it is pgrβ-D_{i-1} space for i =1,2

Proof: This is obvious from definitions 2.1 and 3.5

Theorem 3.7: For a space G, the following assertions are accurate:

(i) G is pgrβ-D_0 space imply and implies G is pgrβ-T_0 space.

(ii) G is pgrβ-D_i space imply and implies G is pgrβ-D_2 space.

Proof: Theorem 4.6 leads to sufficiency for (i) and (ii).

Necessity for (i): Let G be pgrβ-D_0 space so that for each set of two separate elements a and b of G, at least one lies in a pgrβ-D set F. Therefore, we choose a ∈ F and b ∉ F. Suppose F = U \V for U ≠ G for pgrβ-open sets U and V. This suggests that a ∈ U. The argument being ,b ∉ F we have either (a) b ∉ U or (b) b ∈U and b ∈V. For (a) the space G is pgrβ-T_0 since a ∈U and b ∉ U. For (b), the space G is also pgrβ-T_0 since b ∈V but a ∉V.

Necessity for (ii): Suppose G is a pgrβ-D set. It is evident from the definition that for each set of two separate elements a and b in G there are pgrβ-D sets R and S such that R with a but no b and S without a but with b .Let R = U \V and S = W \ Z, where U,V, W and Z are pgrβ-open sets in G. Due to the fact that a ∉S, we have two cases, i.e. either (a) a ∉W or both W and Z contain a. If a ∉W, then from b ∉R either (a) b ∉U or (b) b ∈U and b ∈V.

If (a) is the case, therefore it follows from a ∈ U \V that a ∈ U \ (V ∪W) and also therefore it follows from b ∈W \ Z that b ∈W \ (U ∪Z). Thus we have U \ (V ∪W) and W \ (U ∪ Z) are disjoint.

If (b) is the case, it follows from there that a ∈U \V and b ∈V since b ∈U and b ∈V. Therefore (U\V) ∩V = ∅. If a ∈W and a ∈ Z, we have y ∈W \ Z and a ∈ Z. Hence (W \ Z) ∩ Z = ∅. This demonstrates that G is pgrβ -D_2 space.

Corollary 3.8: If G is pgrβ-D_1 space, then it is pgrβ-T_0 space.

Conclusion

In this research, pgrβ-open sets in topological spaces are used to provide a new type of separation axioms. Additionally, the ideas of pgrβ-T_0, pgrβ-T_1, and pgrβ-T_2 spaces, pgrβ-D set, pgrβ-D_0, pgrβ-D_1, and pgrβ-D_2 spaces are addressed. Some characterizations for these spaces and their relationships with one another have also been looked at.

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